Extra tutorial: Selected problems of Assignment 13

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Recall the notion of radius of convergence:

Def: Given a power series $f(x) := \sum_{n=0}^{\infty} a_n (x-x_0)^n$, let $\rho := \overline{\lim_n} |a_n|^{\frac{1}{n}} \in [0,+\infty]$, the radius of convergence is defined as

$$R := \begin{cases} 0, & \rho = +\infty \\ \frac{1}{\rho}, & 0 < \rho < +\infty \\ \infty, & \rho = 0 \end{cases}$$

The following theorem justifies the name "radius of convergence"

Thm (Cauchy-Hadamard Theorem)

With notations as above, we have the following cases:

a) $R=0$: $f(x)$ diverges on $\mathbb{R} \setminus \{x_0\}$

b) $0 < R < +\infty$: $f(x)$ converges uniformly on every $[a,b] \subseteq (x_0-r, x_0+r)$ and diverges on $|x-x_0| > R$

c) $R = +\infty$: $f(x)$ converges uniformly on every $[a,b] \subseteq \mathbb{R}$

In Q1, Q2, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ (i.e. $x_0 = 0$)

Q1) (§9.4 Q5)

(a) Suppose $L := \lim_n \left| \frac{a_n}{a_{n+1}} \right|$ exists in $[0, +\infty]$, show that $R = L$.

(b) Give an example which f has $R > 0$, but L does not exist in $[0, +\infty]$

Sol: (a) For each $x \in \mathbb{R}$, let $\lambda(x) := \lim_n \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = \lim_n \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = \frac{|x|}{L}$

Case 1: $L = 0$: then for all $x \neq 0$, $\lambda(x) = \infty$.

$\therefore$ By ratio test, $f(x)$ diverges on $\mathbb{R} \setminus \{0\}$.

$\therefore$ By C-H Thm, $R = 0 = L$

Case 2: $0 < L < +\infty$: then for all x with $|x| < L$, $\lambda(x) < 1$.

$\therefore$ By ratio test, $f(x)$ converges absolutely.

On the other hand, for all x with $|x| > L$, $\lambda(x) > 1$

$\therefore$ By ratio test, $f(x)$ diverges. $\therefore$ By C-H Thm, $R = L$.

Case 3 : $L = \infty$ then for all $x \in \mathbb{R}$, $\lambda(x) = 0$

$\therefore$ By ratio test, $f(x)$ converges absolutely.

$\therefore$ By C-H Thm, $R = \infty = L$.

(b) Consider $f(x) = 1 + x^2 + x^4 + \cdots$, i.e.

$$a_n = \begin{cases} 1, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Then $\rho = \overline{\lim_n} |a_n|^{\frac{1}{n}} = 1$. $\therefore R = 1$ is positive.

However, as $a_n = 0$ when n is odd, L does not exist in $[0, +\infty]$.

Q2) (§9.4 Q6a, 6c) Determine R when

(a) $a_n = \frac{1}{n^n}$ (b) $a_n = \frac{n^n}{n!}$

Sol: (a) $\rho = \overline{\lim_n} |a_n|^{\frac{1}{n}} = \overline{\lim_n} \frac{1}{n} = 0$. $\therefore R = \infty$

(b) Try to compute $L = \lim_n \left|\frac{a_n}{a_{n+1}}\right|$:

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{\frac{n^n}{n!}}{\frac{(n+1)^{n+1}}{(n+1)!}} = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{\left(\frac{n+1}{n}\right)}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$\therefore L = \lim_n \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$.

$\therefore$ By (Q1a), $R = L = \frac{1}{e}$.

Q3) (§9.4 Q11)

Let $f:(-r,r)\to\mathbb{R}$ be a smooth function

such that $\exists B>0$, $\forall n\in\mathbb{N}$, $\|f^{(n)}\|_\infty\leq B$.

Show that $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ converges uniformly to $f(x)$ on $(-r,r)$

Sol: Since $\lim_k \frac{r^{k+1}}{(k+1)!}=0$ (by applying nth term test to $\sum_{k=0}^{\infty}\frac{r^{k+1}}{(k+1)!}$)

Given $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that for any $k\geq K$

$$\frac{r^{k+1}}{(k+1)!}<\frac{\varepsilon}{B}$$

then for any $x\in(-r,r)$, for any $k\geq K$,

by Taylor's Thm on f with $x_0=0$, there exists c with

$0<|c|<|x|$ such that $f(x)=\sum_{n=0}^{k}\frac{f^{(n)}(0)}{n!}x^n+\frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$.

$$\therefore \left|f(x)-\sum_{n=0}^{k}\frac{f^{(n)}(0)}{n!}x^n\right|=\left|\frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}\right|\leq\frac{B}{(k+1)!}r^{k+1}<\varepsilon$$

$\therefore \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ converges uniformly to f on $(-r,r)$

Q4) (Supp. Ex. 1)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series at 0 with $R(f) > 0$.

(a) Show that for each $k \in \mathbb{N} \cup \{0\}$, $a_k = \frac{f^{(k)}(0)}{k!}$

(b) If $g(x) = \sum_{n=0}^{\infty} b_n x^n$ is another power series at 0 with $R(g) > 0$

so that there exists $r > 0$ such that $f = g$ on $(-r, r)$,

show that for all $k \in \mathbb{N} \cup \{0\}$, $a_k = b_k$.

Sol: (a) For $k = 0$, substituting $x = 0$ gives $f(0) = a_0$

For $k > 0$, applying Differentiation Theorem k times,

$f^{(k)}(x) = \sum_{n=k}^{\infty} a_n \cdot (n(n-1)\cdots(n-k+1))\, x^{n-k}$ satisfies $R(f^{(k)}) = R(f)$

$\therefore$ Substituting $x = 0$, $f^{(k)}(0) = a_k \cdot k!$, $\therefore a_k = \frac{f^{(k)}(0)}{k!}$

(b) For each $k \in \mathbb{N} \cup \{0\}$, applying (a) to f and g gives

$a_k = \frac{f^{(k)}(0)}{k!}$ and $b_k = \frac{g^{(k)}(0)}{k!}$. As $f = g$ on $(-r, r)$,

$f^{(k)}(0) = g^{(k)}(0)$, $\therefore a_k = \frac{f^{(k)}(0)}{k!} = \frac{g^{(k)}(0)}{k!} = b_k$